Pi Formulas

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There are many formulas of π of many types. Among others, these include series, products, geometric constructions, limits, special values, and pi iterations.

π is intimately related to the properties of circles and spheres. For a circle of radius r, the circumference and area are given by

$$C = 2\pi r \quad (1)$$
$$A = \pi r^2. \quad (2)$$

Similarly, for a sphere of radius r, the surface area and volume enclosed are

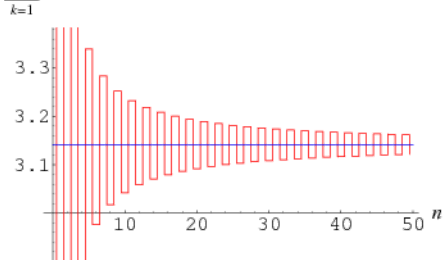
$$S = 4\pi r^2 \quad (3)$$
$$V = \tfrac{4}{3}\pi r^3. \quad (4)$$

An exact formula for π in terms of the inverse tangents of unit fractions is Machin's formula

$$\tfrac{1}{4}\pi = 4\tan^{-1}\left(\tfrac{1}{5}\right) - \tan^{-1}\left(\tfrac{1}{239}\right). \quad (5)$$

There are three other Machin-like formulas, as well as thousands of other similar formulas having more terms.

$$4\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}$$



Gregory and Leibniz found

$$\frac{\pi}{4} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1} \quad (6)$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \ldots \quad (7)$$

(Wells 1986, p. 50), which is known as the Gregory series and may be obtained by plugging $x = 1$ into the Leibniz series for $\tan^{-1} x$. The error after the nth term of this series in the Gregory series is larger than $(2n)^{-1}$ so this sum converges so slowly that 300 terms are not sufficient to calculate π correctly to two decimal places! However, it can be transformed to

$$\pi = \sum_{k=1}^{\infty}\frac{3^k - 1}{4^k}\zeta(k+1), \quad (8)$$

where $\zeta(z)$ is the Riemann zeta function (Vardi 1991, pp. 157-158; Flajolet and Vardi 1996), so that the error after k terms is $\approx (3/4)^k$.

An infinite sum series to Abraham Sharp (ca. 1717) is given by

$$\pi = \sum_{k=0}^{\infty}\frac{2(-1)^k 3^{1/2-k}}{2k+1} \quad (9)$$

(Smith 1953, p. 311). Additional simple series in which π appears are

$$\tfrac{1}{4}\pi\sqrt{2} = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}}{4k-1} + \frac{(-1)^{k+1}}{4k-3}\right] \quad (10)$$
$$= 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \ldots \quad (11)$$
$$\tfrac{1}{4}(\pi - 3) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k(2k+1)(2k+2)} \quad (12)$$
$$= \frac{1}{2\cdot 3\cdot 4} - \frac{1}{4\cdot 5\cdot 6} + \frac{1}{6\cdot 7\cdot 8} - \ldots \quad (13)$$
$$\tfrac{1}{6}\pi^2 = \sum_{k=1}^{\infty}\frac{1}{k^2} \quad (14)$$
$$= 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \ldots \quad (15)$$
$$\tfrac{1}{8}\pi^2 = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \quad (16)$$
$$= \quad (17)$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$$

(Wells 1986, p. 53).

In 1666, Newton used a geometric construction to derive the formula

$$\pi = \tfrac{3}{4}\sqrt{3}+24\int_0^{1/4}\sqrt{x-x^2}\,dx \tag{18}$$

$$= \frac{3\sqrt{3}}{4}+24\left(\frac{1}{12}-\frac{1}{5\cdot 2^5}-\frac{1}{28\cdot 2^7}-\frac{1}{72\cdot 2^9}-\ldots\right), \tag{19}$$

which he used to compute π (Wells 1986, p. 50; Borwein *et al.* 1989; Borwein and Bailey 2003, pp. 105-106). The coefficients can be found from the integral

$$I(x) = \int\sqrt{x-x^2}\,dx \tag{20}$$

$$= \tfrac{1}{4}(2x-1)\sqrt{x-x^2}-\tfrac{1}{8}\sin^{-1}(1-2x) \tag{21}$$

by taking the series expansion of $I(x)-I(0)$ about 0, obtaining

$$I(x) = \tfrac{2}{3}x^{3/2}-\tfrac{1}{5}x^{5/2}-\tfrac{1}{28}x^{7/2}-\tfrac{1}{72}x^{9/2}-\tfrac{5}{704}x^{11/2}+\ldots \tag{22}$$

(OEIS A054387 and A054388). Using Euler's convergence improvement transformation gives

$$\frac{\pi}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(n!)^2\,2^{n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{n!}{(2n+1)!!} \tag{23}$$

$$= 1+\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\ldots \tag{24}$$

$$= 1+\frac{1}{3}\left(1+\frac{2}{5}\left(1+\frac{3}{7}\left(1+\frac{4}{9}(1+\ldots)\right)\right)\right) \tag{25}$$

(Beeler *et al.* 1972, Item 120).

This corresponds to plugging $x=1/\sqrt{2}$ into the power series for the hypergeometric function ${}_2F_1(a,b;c;x)$,

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = \sum_{i=0}^{\infty}\frac{(2x)^{2i+1}\,i!^2}{2(2i+1)!} = {}_2F_1\left(1,1;\tfrac{3}{2};x^2\right)x. \tag{26}$$

Despite the convergence improvement, series (◇) converges at only one bit/term. At the cost of a square root, Gosper has noted that $x=1/2$ gives 2 bits/term,

$$\tfrac{1}{9}\sqrt{3}\,\pi = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(i!)^2}{(2i+1)!}, \tag{27}$$

and $x=\sin(\pi/10)$ gives almost 3.39 bits/term,

$$\frac{\pi}{5\sqrt{\phi+2}} = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(i!)^2}{\phi^{2i+1}(2i+1)!}, \tag{28}$$

where ϕ is the golden ratio. Gosper also obtained

$$\pi = 3+\frac{1}{60}\left(8+\frac{2\cdot 3}{7\cdot 8\cdot 3}\left(13+\frac{3\cdot 5}{10\cdot 11\cdot 3}\left(18+\frac{4\cdot 7}{13\cdot 14\cdot 3}(23+\ldots)\right)\right)\right). \tag{29}$$

A spigot algorithm for π is given by Rabinowitz and Wagon (1995; Borwein and Bailey 2003, pp. 141-142).

More amazingly still, a closed form expression giving a digit-extraction algorithm which produces digits of π (or π^2) in base-16 was discovered by Bailey *et al.* (Bailey *et al.* 1997, Adamchik and Wagon 1997),

$$\pi = \sum_{n=0}^{\infty}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)\left(\frac{1}{16}\right)^n. \tag{30}$$

This formula, known as the BBP formula, was discovered using the PSLQ algorithm (Ferguson *et al.* 1999) and is equivalent to

$$\pi = \int_0^1\frac{16y-16}{y^4-2y^3+4y-4}\,dy. \tag{31}$$

There is a series of BBP-type formulas for π in powers of $(-1)^k$, the first few independent formulas of which are

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \tag{32}$$

$$= 3\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{6k+1}+\frac{1}{6k+5}\right] \tag{33}$$

$$= 4\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{10k+1}-\frac{1}{10k+3}+\frac{1}{10k+5}-\frac{1}{10k+7}+\frac{1}{10k+9}\right] \tag{34}$$

$$= \sum_{k=0}^{\infty}(-1)^k\left[\frac{3}{14k+1}-\frac{3}{14k+3}+\frac{3}{14k+5}+\frac{4}{14k+7}+\frac{4}{14k+9}-\frac{4}{14k+11}+\frac{4}{14k+13}\right] \tag{35}$$

$$= \sum_{k=0}^{\infty}(-1)^k\left[\frac{2}{18k+1}+\frac{3}{18k+3}+\frac{2}{18k+5}-\frac{2}{18k+7}-\frac{2}{18k+11}+\frac{2}{18k+13}+\frac{3}{18k+15}+\frac{2}{18k+17}\right] \tag{36}$$

$$= \tag{37}$$

$$\sum_{k=0}^{\infty}(-1)^k\left[\frac{3}{22k+1}-\frac{3}{22k+3}+\frac{3}{22k+5}-\frac{3}{22k+7}+\frac{3}{22k+9}+\frac{8}{22k+11}+\frac{3}{22k+13}-\frac{3}{22k+15}+\frac{3}{22k+17}-\frac{3}{22k+19}+\frac{1}{22k+21}\right].$$

Similarly, there are a series of BBP-type formulas for π in powers of 2^k, the first few independent formulas of which are

$$\pi=\sum_{k=0}^{\infty}\frac{1}{16^k}\left[\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right] \tag{38}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{16^k}\left[\frac{8}{8k+2}+\frac{4}{8k+3}+\frac{4}{8k+4}-\frac{1}{8k+7}\right] \tag{39}$$

$$=\frac{1}{16}\sum_{k=0}^{\infty}\frac{1}{256^k}\left[\frac{64}{16k+1}-\frac{32}{16k+4}-\frac{16}{16k+5}-\frac{16}{16k+6}+\frac{4}{16k+9}-\frac{2}{16k+12}-\frac{1}{16k+13}-\frac{1}{16k+14}\right] \tag{40}$$

$$=\frac{1}{32}\sum_{k=0}^{\infty}\frac{1}{256^k}\left[\frac{128}{16k+2}+\frac{64}{16k+3}+\frac{64}{16k+4}-\frac{16}{16k+7}+\frac{8}{16k+10}+\frac{4}{16k+11}+\frac{4}{16k+12}-\frac{1}{16k+15}\right] \tag{41}$$

$$=\frac{1}{32}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left[\frac{256}{24k+2}+\frac{192}{24k+3}-\frac{256}{24k+4}-\frac{96}{24k+6}-\frac{96}{24k+8}+\frac{16}{24k+10}-\frac{4}{24k+12}-\frac{3}{24k+15}-\frac{6}{24k+16}-\frac{2}{24k+18}-\frac{1}{24k+20}\right] \tag{42}$$

$$=\frac{1}{64}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left[\frac{256}{24k+1}+\frac{256}{24k+2}-\frac{384}{24k+3}-\frac{256}{24k+4}-\frac{64}{24k+5}+\frac{96}{24k+8}+\frac{64}{24k+9}+\frac{16}{24k+10}+\frac{8}{24k+12}-\frac{4}{24k+13}+\frac{6}{24k+15}+\frac{6}{24k+16}+\frac{1}{24k+17}+\frac{1}{24k+18}-\frac{1}{24k+20}-\frac{1}{24k+21}\right] \tag{43}$$

$$=\frac{1}{96}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left[\frac{256}{24k+2}+\frac{64}{24k+3}+\frac{128}{24k+5}+\frac{352}{24k+6}+\frac{64}{24k+7}+\frac{288}{24k+8}+\frac{128}{24k+9}+\frac{80}{24k+10}+\frac{20}{24k+12}-\frac{16}{24k+14}-\frac{1}{24k+15}+\frac{6}{24k+16}-\frac{2}{24k+17}-\frac{1}{24k+19}+\frac{1}{24k+20}-\frac{2}{24k+21}\right] \tag{44}$$

$$=\frac{1}{96}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left[\frac{256}{24k+1}+\frac{320}{24k+3}+\frac{256}{24k+4}-\frac{192}{24k+5}-\frac{224}{24k+6}-\frac{64}{24k+7}-\frac{192}{24k+8}-\frac{64}{24k+9}-\frac{64}{24k+10}-\frac{28}{24k+12}-\frac{4}{24k+13}-\frac{5}{24k+15}+\frac{3}{24k+17}+\frac{1}{24k+18}+\frac{1}{24k+19}+\frac{1}{24k+21}-\frac{1}{24k+22}\right] \tag{45}$$

$$=\frac{1}{96}\sum_{k=0}^{\infty}\frac{1}{4096^k}\left[\frac{512}{24k+1}-\frac{256}{24k+2}+\frac{64}{24k+3}-\frac{512}{24k+4}-\frac{32}{24k+6}+\frac{64}{24k+7}+\frac{96}{24k+8}+\frac{64}{24k+9}+\frac{48}{24k+10}-\frac{12}{24k+12}-\frac{8}{24k+13}-\frac{16}{24k+14}-\frac{1}{24k+15}-\frac{6}{24k+16}-\frac{2}{24k+18}-\frac{1}{24k+19}-\frac{1}{24k+20}-\frac{1}{24k+21}\right] \tag{46}$$

$$=\frac{1}{4096}\sum_{k=0}^{\infty}\frac{1}{65\,536^k}\left[\frac{16\,384}{32k+1}-\frac{8192}{32k+4}-\frac{4096}{32k+5}-\frac{4096}{32k+6}+\frac{1024}{32k+9}-\frac{512}{32k+12}-\frac{256}{32k+13}-\frac{256}{32k+14}+\frac{64}{32k+17}-\frac{32}{32k+20}-\frac{16}{32k+21}-\frac{16}{32k+22}+\frac{4}{32k+25}-\frac{2}{32k+28}-\frac{1}{32k+29}-\frac{1}{32k+30}\right] \tag{47}$$

$$=\frac{1}{4096}\sum_{k=0}^{\infty}\frac{1}{65\,536^k}\left[\frac{32\,768}{32k+2}+\frac{16\,384}{32k+3}+\frac{16\,384}{32k+4}-\frac{4096}{32k+7}+\frac{2048}{32k+10}+\frac{1024}{32k+11}+\frac{1024}{32k+12}-\frac{256}{32k+15}+\frac{128}{32k+18}+\frac{64}{32k+19}+\frac{64}{32k+20}-\frac{16}{32k+23}+\frac{8}{32k+26}+\frac{4}{32k+27}+\frac{4}{32k+28}-\frac{1}{32k+31}\right]. \tag{48}$$

F. Bellard found the rapidly converging BBP-type formula

$$\pi=\frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right). \tag{49}$$

A related integral is

$$\pi=\frac{22}{7}-\int_0^1\frac{x^4(1-x)^4}{1+x^2}\,dx \tag{50}$$

(Dalzell 1944, 1971; Le Lionnais 1983, p. 22; Borwein, Bailey, and Girgensohn 2004, p. 3; Boros and Moll 2004, p. 125; Lucas 2005; Borwein *et al.* 2007, p. 14). This integral was known by K. Mahler in the mid-1960s and appears in an exam at the University of Sydney in November 1960 (Borwein, Bailey, and Girgensohn, p. 3). Beukers (2000) and Boros and Moll (2004, p. 126) state that it is not clear if these exists a natural choice of rational polynomial whose integral between 0 and 1 produces $\pi - 333/106$, where 333/106 is the next convergent. However, an integral exists for the *fourth* convergent, namely

$$\pi = \frac{355}{113} - \frac{1}{3164} \int_0^1 \frac{x^8 (1-x)^8 \left(25 + 816 x^2\right)}{1+x^2} \, d x. \qquad (51)$$

(Lucas 2005; Bailey *et al.* 2007, p. 219). In fact, Lucas (2005) gives a few other such integrals.

Backhouse (1995) used the identity

$$I_{m,n} = \int_0^1 \frac{x^m (1-x)^n}{1+x^2} \, d x \qquad (52)$$

$$= 2^{-(m+n+1)} \sqrt{\pi} \, \Gamma(m+1) \Gamma(n+1) \times {}_3F_2\left(1, \frac{m+1}{2}, \frac{m+2}{2}; \frac{m+n+2}{2}, \frac{m+n+3}{2}; -1\right) \qquad (53)$$

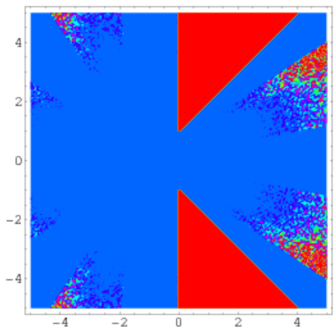
$$= a + b\pi + c \ln 2 \qquad (54)$$

for positive integer m and n and where a, b, and c are rational constant to generate a number of formulas for π. In particular, if $2m - n \equiv 0 \pmod 4$, then $c = 0$ (Lucas 2005).

A similar formula was subsequently discovered by Ferguson, leading to a two-dimensional lattice of such formulas which can be generated by these two formulas given by

$$\pi = \sum_{k=0}^{\infty} \left(\frac{4+8r}{8k+1} - \frac{8r}{8k+2} - \frac{4r}{8k+3} - \frac{2+8r}{8k+4} - \frac{1+2r}{8k+5} - \frac{1+2r}{8k+6} + \frac{r}{8k+7}\right) \left(\frac{1}{16}\right)^k \qquad (55)$$

for any complex value of r (Adamchik and Wagon), giving the BBP formula as the special case $r = 0$.



An even more general identity due to Wagon is given by

$$\pi + 4 \tan^{-1} z + 2 \ln\left(\frac{1 - 2z - z^2}{z^2 + 1}\right) = \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{4(z+1)^{8k+1}}{8k+1} - \frac{2(z+1)^{8k+4}}{8k+4} - \frac{(z+1)^{8k+5}}{8k+5} - \frac{(z+1)^{8k+6}}{8k+6}\right] \qquad (56)$$

(Borwein and Bailey 2003, p. 141), which holds over a region of the complex plane excluding two triangular portions symmetrically placed about the real axis, as illustrated above.

A perhaps even stranger general class of identities is given by

$$\pi = 4 \sum_{j=1}^{n} \frac{(-1)^{j+1}}{2j-1} + \frac{(-1)^n (2n-1)!}{4} \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{8}{(8k+1)_{2n}} - \frac{4}{(8k+3)_{2n}} - \frac{4}{(8k+4)_{2n}} - \frac{2}{(8k+5)_{2n}} + \frac{1}{(8k+7)_{2n}} + \frac{1}{(8k+8)_{2n}}\right] \qquad (57)$$

which holds for any positive integer n, where $(x)_n$ is a Pochhammer symbol (B. Cloitre, pers. comm., Jan. 23, 2005). Even more amazingly, there is a closely analogous formula for the natural logarithm of 2.

Following the discovery of the base-16 digit BBP formula and related formulas, similar formulas in other bases were investigated. Borwein, Bailey, and Girgensohn (2004) have recently shown that π has no Machin-type BBP arctangent formula that is not binary, although this does not rule out a completely different scheme for digit-extraction algorithms in other bases.

S. Plouffe has devised an algorithm to compute the nth digit of π in any base in $O\left(n^3 (\log n)^3\right)$ steps.

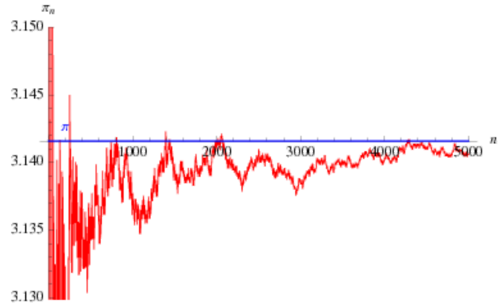
A slew of additional identities due to Ramanujan, Catalan, and Newton are given by Castellanos (1988ab, pp. 86-88), including several involving sums of Fibonacci numbers. Ramanujan found

$$\sum_{k=0}^{\infty} \frac{(-1)^k (4k+1) [(2k-1)!!]^3}{[(2k)!!]^3} = \sum_{k=0}^{\infty} \frac{(-1)^k (4k+1) \left[\Gamma\left(k+\frac{1}{2}\right)\right]^3}{\pi^{3/2} [\Gamma(k+1)]^3} = \frac{2}{\pi} \qquad (58)$$

(Hardy 1923, 1924, 1999, p. 7).

Plouffe (2006) found the beautiful formula

$$\pi = 72 \sum_{n=1}^{\infty} \frac{1}{n\left(e^{n\pi} - 1\right)} - 96 \sum_{n=1}^{\infty} \frac{1}{n\left(e^{2n\pi} - 1\right)} + 24 \sum_{n=1}^{\infty} \frac{1}{n\left(e^{4n\pi} - 1\right)}. \qquad (59)$$



An interesting infinite product formula due to Euler which relates π and the nth prime p_n is

$$\pi = \frac{2}{\prod_{n=1}^{\infty}\left[1+\frac{\sin\left(\frac{1}{2}\pi p_n\right)}{p_n}\right]} \quad (60)$$

$$= \frac{2}{\prod_{n=2}^{\infty}\left[1+\frac{(-1)^{(p_n-1)/2}}{p_n}\right]} \quad (61)$$

(Blatner 1997, p. 119), plotted above as a function of the number of terms in the product.

A method similar to Archimedes' can be used to estimate π by starting with an n-gon and then relating the area of subsequent $2\,n$-gons. Let β be the angle from the center of one of the polygon's segments,

$$\beta = \tfrac{1}{4}(n-3)\pi, \quad (62)$$

then

$$\pi = \frac{2\sin(2\beta)}{(n-3)\prod_{k=0}^{\infty}\cos\left(2^{-k}\beta\right)} \quad (63)$$

(Beckmann 1989, pp. 92-94).

Vieta (1593) was the first to give an exact expression for π by taking $n=4$ in the above expression, giving

$$\cos\beta = \sin\beta = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}, \quad (64)$$

which leads to an infinite product of nested radicals,

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\;\cdots \quad (65)$$

(Wells 1986, p. 50; Beckmann 1989, p. 95). However, this expression was not rigorously proved to converge until Rudio in 1892.

A related formula is given by

$$\pi = \lim_{n\to\infty} 2^{n+1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}\,, \quad (66)$$

which can be written

$$\pi = \lim_{n\to\infty} 2^{n+1}\pi_n, \quad (67)$$

where π_n is defined using the iteration

$$\pi_n = \sqrt{\left(\tfrac{1}{2}\pi_{n-1}\right)^2 + \left[1-\sqrt{1-\left(\tfrac{1}{2}\pi_{n-1}\right)^2}\right]^2} \quad (68)$$

with $\pi_0 = \sqrt{2}$ (J. Munkhammar, pers. comm., April 27, 2000). The formula

$$\pi = 2\lim_{m\to\infty}\sum_{n=1}^{m}\sqrt{\left[\sqrt{1-\left(\frac{n-1}{m}\right)^2}-\sqrt{1-\left(\frac{n}{m}\right)^2}\right]^2+\frac{1}{m^2}} \quad (69)$$

is also closely related.

A pretty formula for π is given by

$$\pi = \frac{\prod_{n=1}^{\infty}\left(1+\frac{1}{4n^2-1}\right)}{\sum_{n=1}^{\infty}\frac{1}{4n^2-1}}, \quad (70)$$

where the numerator is a form of the Wallis formula for $\pi/2$ and the denominator is a telescoping sum with sum 1/2 since

$$\frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) \quad (71)$$

(Sondow 1997).

A particular case of the Wallis formula gives

(72)

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] = \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots$$

(Wells 1986, p. 50). This formula can also be written

$$\lim_{n\to\infty}\frac{2^{4n}}{n\binom{2n}{n}^2} = \pi\lim_{n\to\infty}\frac{n\,[\Gamma(n)]^2}{\left[\Gamma\left(\frac{1}{2}+n\right)\right]^2} = \pi, \qquad (73)$$

where $\binom{n}{k}$ denotes a binomial coefficient and $\Gamma(x)$ is the gamma function (Knopp 1990). Euler obtained

$$\pi = \sqrt{6\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)}, \qquad (74)$$

which follows from the special value of the Riemann zeta function $\zeta(2) = \pi^2/6$. Similar formulas follow from $\zeta(2n)$ for all positive integers n.

An infinite sum due to Ramanujan is

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{42n+5}{2^{12n+4}} \qquad (75)$$

(Borwein *et al.* 1989; Borwein and Bailey 2003, p. 109; Bailey *et al.* 2007, p. 44). Further sums are given in Ramanujan (1913-14),

$$\frac{4}{\pi} = \sum_{n=0}^{\infty}\frac{(-1)^n(1123+21\,460\,n)(2n-1)!!\,(4n-1)!!}{882^{2n+1}\,32^n\,(n!)^3} \qquad (76)$$

and

$$\frac{1}{\pi} = \sqrt{8}\sum_{n=0}^{\infty}\frac{(1103+26\,390\,n)(2n-1)!!\,(4n-1)!!}{99^{4n+2}\,32^n\,(n!)^3} \qquad (77)$$

$$= \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26\,390\,n)}{(n!)^4\,396^{4n}} \qquad (78)$$

(Beeler *et al.* 1972, Item 139; Borwein *et al.* 1989; Borwein and Bailey 2003, p. 108; Bailey *et al.* 2007, p. 44). Equation (78) is derived from a modular identity of order 58, although a first derivation was not presented prior to Borwein and Borwein (1987). The above series both give

$$\pi \approx \frac{9801}{2206\sqrt{2}} = 3.14159273001\ldots \qquad (79)$$

(Wells 1986, p. 54) as the first approximation and provide, respectively, about 6 and 8 decimal places per term. Such series exist because of the rationality of various modular invariants.

The general form of the series is

$$\sum_{n=0}^{\infty}[a(t)+n\,b(t)]\,\frac{(6n)!}{(3n)!\,(n!)^3}\,\frac{1}{[j(t)]^n} = \frac{\sqrt{-j(t)}}{\pi}, \qquad (80)$$

where t is a binary quadratic form discriminant, $j(t)$ is the j-function,

$$b(t) = \sqrt{t\,[1728-j(t)]} \qquad (81)$$

$$a(t) = \frac{b(t)}{6}\left\{1-\frac{E_4(t)}{E_6(t)}\left[E_2(t)-\frac{6}{\pi\sqrt{t}}\right]\right\}, \qquad (82)$$

and the E_i are Eisenstein series. A class number p field involves pth degree algebraic integers of the constants $A = a(t)$, $B = b(t)$, and $C = c(t)$. Of all series consisting of only integer terms, the one gives the most numeric digits in the shortest period of time corresponds to the largest class number 1 discriminant of $d = -163$ and was formulated by the Chudnovsky brothers (1987). The 163 appearing here is the same one appearing in the fact that $e^{\pi\sqrt{163}}$ (the Ramanujan constant) is very nearly an integer. Similarly, the factor $640\,320^3$ comes from the j-function identity for $j\left(\frac{1}{2}\left(1+i\sqrt{163}\right)\right)$. The series is given by

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!\,(13\,591\,409+545\,140\,134\,n)}{(n!)^3\,(3n)!\,\left(640\,320^3\right)^{n+1/2}} \qquad (83)$$

$$= \frac{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}{640\,320^{3/2}}\sum_{n=0}^{\infty}\left(\frac{13\,591\,409}{163\cdot 2\cdot 9\cdot 7\cdot 11\cdot 19\cdot 127}+n\right)\frac{(6n)!}{(3n)!\,(n!)^3}\,\frac{(-1)^n}{640\,320^{3n}} \qquad (84)$$

(Borwein and Borwein 1993; Beck and Trott; Bailey *et al.* 2007, p. 44). This series gives 14 digits accurately per term. The same equation in another form was given by the Chudnovsky brothers (1987) and is used by the Wolfram Language to calculate π (Vardi 1991; Wolfram Research),

$$\pi = \frac{426\,880\sqrt{10\,005}}{A\left[{}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;B\right)-C\,{}_3F_2\left(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;B\right)\right]}, \qquad (85)$$

where

$$A \equiv 13\,591\,409 \qquad (86)$$

$$B \equiv -\frac{1}{151\,931\,373\,056\,000} \qquad (87)$$

$$C \equiv \frac{30\,285\,563}{1\,651\,969\,144\,908\,540\,723\,200}. \qquad (88)$$

The best formula for class number 2 (largest discriminant -427) is

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!\,(A+Bn)}{(n!)^3\,(3n)!\,C^{n+1/2}}, \qquad (89)$$

where

$$A \equiv 212\,175\,710\,912\sqrt{61}+1\,657\,145\,277\,365 \qquad (90)$$

$$B \equiv \qquad (91)$$

$$C \equiv \left[5280\,(236\,674 + 30\,303\,\sqrt{61})\right]^3 \tag{92}$$

(Borwein and Borwein 1993). This series adds about 25 digits for each additional term. The fastest converging series for class number 3 corresponds to $d = -907$ and gives 37-38 digits per term. The fastest converging class number 4 series corresponds to $d = -1555$ and is

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty} \frac{(6\,n)!}{(3\,n)!\,(n!)^3} \frac{A + n\,B}{C^{3\,n}}, \tag{93}$$

where

$$A = \begin{array}{l} 63\,365\,028\,312\,971\,999\,585\,426\,220 + 28\,337\,702\,140\,800\,842\,046\,825\,600\,\sqrt{5} + \\ 384\,\sqrt{5}\,(10\,891\,728\,551\,171\,178\,200\,467\,436\,212\,395\,209\,160\,385\,656\,017 + \\ \quad 4\,870\,929\,086\,578\,810\,225\,077\,338\,534\,541\,688\,721\,351\,255\,040\,\sqrt{5})^{1/2} \end{array} \tag{94}$$

$$B = \begin{array}{l} 7\,849\,910\,453\,496\,627\,210\,289\,749\,000 + 3\,510\,586\,678\,260\,932\,028\,965\,606\,400\,\sqrt{5} + \\ 2\,515\,968\,\sqrt{3110}\,(6\,260\,208\,323\,789\,001\,636\,993\,322\,654\,444\,020\,882\,161 + \\ \quad 2\,799\,650\,273\,060\,444\,296\,577\,206\,890\,718\,825\,190\,235\,\sqrt{5})^{1/2} \end{array} \tag{95}$$

$$C = \begin{array}{l} -214\,772\,995\,063\,512\,240 - 96\,049\,403\,338\,648\,032 \\ \sqrt{5} - 1296\,\sqrt{5}\,(10\,985\,234\,579\,463\,550\,323\,713\,318\,473 + \\ \quad 4\,912\,746\,253\,692\,362\,754\,607\,395\,912\,\sqrt{5})^{1/2}. \end{array} \tag{96}$$

This gives 50 digits per term. Borwein and Borwein (1993) have developed a general algorithm for generating such series for arbitrary class number.

A complete listing of Ramanujan's series for $1/\pi$ found in his second and third notebooks is given by Berndt (1994, pp. 352-354),

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6\,n+1)\left(\frac{1}{2}\right)_n^3}{4^n\,(n!)^3} \tag{97}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42\,n+5)\left(\frac{1}{2}\right)_n^3}{64^n\,(n!)^3} \tag{98}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \frac{(42\,\sqrt{5}\,n + 5\,\sqrt{5} + 30\,n - 1)\left(\frac{1}{2}\right)_n^3}{64^n\,(n!)^3} \left(\frac{\sqrt{5}-1}{2}\right)^{8\,n} \tag{99}$$

$$\frac{27}{4\,\pi} = \sum_{n=0}^{\infty} \frac{(15\,n+2)\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(n!)^3} \left(\frac{2}{27}\right)^n \tag{100}$$

$$\frac{15\,\sqrt{3}}{2\,\pi} = \sum_{n=0}^{\infty} \frac{(33\,n+4)\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(n!)^3} \left(\frac{4}{125}\right)^n \tag{101}$$

$$\frac{5\,\sqrt{5}}{2\,\pi\,\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(11\,n+1)\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} \left(\frac{4}{125}\right)^n \tag{102}$$

$$\frac{85\,\sqrt{85}}{18\,\pi\,\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(133\,n+8)\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3} \left(\frac{4}{85}\right)^{3\,n} \tag{103}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(20\,n+3)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,2^{2\,n+1}} \tag{104}$$

$$\frac{4}{\pi\,\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(28\,n+3)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,3^n\,4^{2\,n+1}} \tag{105}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(260\,n+23)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,(18)^{2\,n+1}} \tag{106}$$

$$\frac{4}{\pi\,\sqrt{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(644\,n+41)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,5^n\,(72)^{2\,n+1}} \tag{107}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(21\,460\,n+1123)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,(882)^{2\,n+1}} \tag{108}$$

$$\frac{2\,\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \frac{(8\,n+1)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,9^n} \tag{109}$$

$$\frac{1}{2\,\pi\,\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(10\,n+1)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,9^{2\,n+1}} \tag{110}$$

$$\frac{1}{3\,\pi\,\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(40\,n+3)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,(49)^{2\,n+1}} \tag{111}$$

$$\frac{2}{\pi\,\sqrt{11}} = \sum_{n=0}^{\infty} \frac{(280\,n+19)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,(99)^{2\,n+1}} \tag{112}$$

$$\frac{1}{2\,\pi\,\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(26\,390\,n+1103)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\,(99)^{4\,n+2}}. \tag{113}$$

These equations were first proved by Borwein and Borwein (1987a, pp. 177-187). Borwein and Borwein (1987b, 1988, 1993) proved other equations of this type, and Chudnovsky and Chudnovsky (1987) found similar equations for other transcendental constants (Bailey *et al*. 2007, pp. 44-45).

A complete list of independent known equations of this type is given by

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6\,n+1)\left(\frac{1}{2}\right)_n^3}{4^n\,(n!)^3} \tag{114}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42\,n+5)\left(\frac{1}{2}\right)_n^3}{64^n\,(n!)^3} \tag{115}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \frac{\left(42\sqrt{5}\,n+5\sqrt{5}+30\,n-1\right)\left(\frac{1}{2}\right)_n^3}{64^n\,(n!)^3}\left(\frac{\sqrt{5}-1}{2}\right)^{8n} \tag{116}$$

$$\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} \frac{\left(540\sqrt{5}\,n-1200\,n-525+235\sqrt{5}\right)\left(\frac{1}{2}\right)_n^3\left(\sqrt{5}-2\right)^{8n}}{(n!)^3} \tag{117}$$

$$\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} \frac{\left(24\sqrt{3}\,n-36\,n-15+9\sqrt{3}\right)\left(\frac{1}{2}\right)_n^3\left(2-\sqrt{3}\right)^{4n}}{(n!)^3} \tag{118}$$

for $m=1$ with nonalternating signs,

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^3\left(12\sqrt{2}\,n-12\,n-5+4\sqrt{2}\right)\left(\sqrt{2}-1\right)^{4n}}{(n!)^3} \tag{119}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^3\left(60\,n-24\sqrt{5}\,n+23-10\sqrt{5}\right)\left(\sqrt{5}-2\right)^{4n}}{(n!)^3} \tag{120}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^3\left(420\,n-168\sqrt{6}\,n+177-72\sqrt{6}\right)}{(n!)^3} \tag{121}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^3\left(2\sqrt{2}\right)^{2n}}{(n!)^3} \tag{122}$$

for $m=1$ with alternating signs,

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^5\left(820\,n^2+180\,n+13\right)}{32^{2n}\,(n!)^5} \tag{123}$$

$$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}\right)_n^5\left(20\,n^2+8\,n+1\right)}{2^{2n}\,(n!)^5} \tag{124}$$

for $m=2$ (Guillera 2002, 2003, 2006),

$$\frac{32}{\pi^3} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7\left(168\,n^3+76\,n^2+14\,n+1\right)}{32^{2n}\,(n!)^5} \tag{125}$$

for $m=3$ (Guillera 2002, 2003, 2006), and no others for $m>3$ are known (Bailey *et al.* 2007, pp. 45-48).

Bellard gives the exotic formula

$$\pi = \frac{1}{740\,025}\left[\sum_{n=1}^{\infty} \frac{3\,P(n)}{\binom{7\,n}{2\,n}\,2^{n-1}} - 20\,379\,280\right], \tag{126}$$

where

$$P(n) \equiv -885\,673\,181\,n^5 + 3\,125\,347\,237\,n^4 - 2\,942\,969\,225\,n^3 + 1\,031\,962\,795\,n^2 - 196\,882\,274\,n + 10\,996\,648. \tag{127}$$

Gasper quotes the result

$$\pi = \frac{16}{3}\left[\lim_{x\to\infty} x\,{}_1F_2\left(\tfrac{1}{2};2,3;-x^2\right)\right]^{-1}, \tag{128}$$

where ${}_1F_2$ is a generalized hypergeometric function, and transforms it to

$$\pi = \lim_{x\to\infty} 4\,x\,{}_1F_2\left(\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{2};-x^2\right). \tag{129}$$

A fascinating result due to Gosper is given by

$$\lim_{n\to\infty} \prod_{i=n}^{2n} \frac{\pi}{2\tan^{-1} i} = 4^{1/\pi} = 1.554682275\ldots. \tag{130}$$

π satisfies the inequality

$$\left(1+\frac{1}{\pi}\right)^{\pi+1} \approx 3.14097 < \pi. \tag{131}$$

D. Terr (pers. comm.) noted the curious identity

$$(3,1,4) \equiv (1,5,9) + (2,6,5) \pmod{10} \tag{132}$$

involving the first 9 digits of pi.

SEE ALSO:
BBP Formula, Digit-Extraction Algorithm, Pi, Pi Approximations, Pi Continued Fraction, Pi Digits, Pi Iterations, Pi Squared, Spigot Algorithm

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